STRUCTURE OF SEMISIMPLE HOPF ALGEBRAS OF DIMENSION p^2q^2, II

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Abstract. Let k be an algebraically closed field of characteristic 0. In this paper, we obtain the structure theorems for semisimple Hopf algebras of dimension p^2q^2 over k, where p, q are prime numbers with $p^2 < q$. As an application, we also obtain the structure theorems for semisimple Hopf algebras of dimension $9p^2$ and $25q^2$ for all primes $3 \neq p$ and $5 \neq q$.

1. Introduction

Throughout this paper, we will work over an algebraically closed field k of characteristic 0.

Quite recently, an outstanding classification result was obtained for semisimple Hopf algebras over k. That is, Etingof et al [6] completed the classification of semisimple Hopf algebras of dimension pq^2 and pqr, where p, q, r are distinct prime numbers. The results in [6] showed that all these Hopf algebras can be constructed from group algebras and their duals by means of extensions. Up to now, besides those mentioned above, semisimple Hopf algebras of dimension p, p^2, p^3 and pq have been completely classified. See [5, 8, 12, 13, 14, 23] for details.

Recall that a semisimple Hopf algebra H is called of Frobenius type if the dimensions of the simple H-modules divide the dimension of H. Kaplansky conjectured that every finite-dimensional semisimple Hopf algebra is of Frobenius type [9, Appendix 2]. It is still an open problem. Many examples show that a positive answer to Kaplansky's conjecture would be very helpful in the classification of semisimple Hopf algebras. See [3] and the examples mentioned above for details.

In a previous paper [4], we studied the structure of semisimple Hopf algebras of dimension p^2q^2, where p, q are prime numbers with $p^4 < q$. As an application, we also studied the structure of semisimple Hopf algebras of dimension $4q^2$, where q is a prime number. In the present paper, we shall continue our investigation and prove that the main results in [4] can be extended to the case $p^2 < q$. Moreover, the structure theorems for semisimple Hopf algebras of dimension $9p^2$ and $25q^2$ will also be given in this paper, where $3 \neq p$ and $5 \neq q$ are prime numbers.

The paper is organized as follows. In Section 2, we recall the definitions and basic properties of semisolvability, characters and Radford's biproducts, respectively. Some useful lemmas are also obtained in this section. In particular, we give an partial answer to Kaplansky's conjecture. We prove that if dim$H$ is odd and H has a simple module of dimension 3 then 3 divides dim$H$. Under the assumption that H does not have simple modules of dimension 3 and 7, we also prove that if dim$H$ is odd and H has a simple module of dimension 5 then 5 divides dim$H$.

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We begin our main work in Section 3. Let H be a semisimple Hopf algebras of dimension p^2q^2, where $p<q$ is a prime number. We first prove that if $|G(H^*)|=q^2$ then H is upper semisolvable, in the sense of [15]. It is a generalization of [4, Lemma 3.4]. We then present our main result. We prove that if $p^2<q$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order p^2, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2. Our approach is mainly based on looking for normal Hopf subalgebras of H of dimension pq^2. In Section 4, we shall study the structure of semisimple Hopf algebras of dimension $9p^2$ and $25q^2$.

Throughout this paper, all modules and comodules are left modules and left comodules, and moreover they are finite-dimensional over k. $\otimes$, dim mean $\otimes_k$, $\dim_k$, respectively. Our references for the theory of Hopf algebras are [16] or [22]. The notation for Hopf algebras is standard. For example, the group of group-like elements in H is denoted by $G(H)$.

2. Preliminaries

2.1. **Characters.** Throughout this subsection, H will be a semisimple Hopf algebra over k. As an algebra, H is isomorphic to a direct product of full matrix algebras

$$H\cong k^{(n_1)}\times\prod_{i=2}^{s}M_{d_i}(k)^{(n_i)},$$

where $n_1=|G(H^*)|$. In this case, we say H is of type $(d_1,n_1;\cdots;d_s,n_s)$ as an algebra, where $d_1=1$. If H^* is of type $(d_1,n_1;\cdots;d_s,n_s)$ as an algebra, we shall say that H is of type $(d_1,n_1;\cdots;d_s,n_s)$ as a coalgebra.

Obviously, H is of type $(d_1,n_1;\cdots;d_s,n_s)$ as an algebra if and only if H has n_1 non-isomorphic irreducible characters of degree d_1, n_2 non-isomorphic irreducible characters of degree d_2, etc. In this paper, we shall use the notation X_t to denote the set of all irreducible characters of H of degree t.

Let V be an H-module. The character of V is the element $\chi=\chi_V\in H^*$ defined by $\langle\chi,h\rangle=\mathrm{Tr}_V(h)$ for all $h\in H$. The degree of χ is defined to be the integer $\deg\chi=\chi(1)=\dim V$. If U is another H-module, we have

$$\chi_{U\otimes V}=\chi_U\chi_V,\quad \chi_{V^*}=S(\chi_V),$$

where S is the antipode of H^*.

All irreducible characters of H span a subalgebra $R(H)$ of H^*, which is called the character algebra of H. By [23, Lemma 2], $R(H)$ is semisimple. The antipode S induces an anti-algebra involution $*:R(H)\to R(H)$, given by $\chi\mapsto\chi^*:=S(\chi)$. The character of the trivial H-module is the counit ε.

Let $\chi_U,\chi_V\in R(H)$ be the characters of the H-modules U and V, respectively. The integer $m(\chi_U,\chi_V)=\dim\mathrm{Hom}_H(U,V)$ is defined to be the multiplicity of U in V. This can be extended to a bilinear form $m:R(H)\times R(H)\to k$.

Let $\mathrm{Irr}(H)$ denote the set of irreducible characters of H. Then $\mathrm{Irr}(H)$ is a basis of $R(H)$. If $\chi\in R(H)$, we may write $\chi=\sum_{\alpha\in\mathrm{Irr}(H)}m(\alpha,\chi)\alpha$. Let $\chi,\psi,\omega\in R(H)$. Then $m(\chi,\psi\omega)=m(\psi^*,\omega\chi^*)=m(\psi,\chi\omega^*)$ and $m(\chi,\psi)=m(\chi^*,\psi^*)$. See [19, Theorem 9].

For each group-like element g in $G(H^*)$, we have $m(g,\chi\psi)=1$, if $\psi=\chi^*g$ and 0 otherwise for all $\chi,\psi\in\mathrm{Irr}(H)$. In particular, $m(g,\chi\psi)=0$ if $\deg\chi\neq\deg\psi$. Let $\chi\in\mathrm{Irr}(H)$. Then for any group-like element g in $G(H^*)$, $m(g,\chi\chi^*)>0$ if and only

if $m(g,\chi\chi^*)=1$ if and only if $g\chi=\chi$. The set of such group-like elements forms a subgroup of $G(H^*)$, of order at most $(\deg\chi)^2$. See [19, Theorem 10]. Denote this subgroup by $G[\chi]$. In particular, we have

$$\chi\chi^*=\sum_{g\in G[\chi]}g+\sum_{\alpha\in\mathrm{Irr}(H),\deg\alpha>1}m(\alpha,\chi\chi^*)\alpha.$$

A subalgebra A of $R(H)$ is called a standard subalgebra if A is spanned by irreducible characters of H. Let X be a subset of $\mathrm{Irr}(H)$. Then X spans a standard subalgebra of $R(H)$ if and only if the product of characters in X decomposes as a sum of characters in X. There is a bijection between $*$-invariant standard subalgebras of $R(H)$ and quotient Hopf algebras of H. See [19, Theorem 6].

In the rest of this subsection, we shall present some results on irreducible characters and algebra types.

Lemma 2.1. *Let $\chi\in\mathrm{Irr}(H)$ be an irreducible character of H. Then*

(1) The order of $G[\chi]$ divides $(\deg\chi)^2$.

(2) The order of $G(H^*)$ divides $n(\deg\chi)^2$, where n is the number of non-isomorphic irreducible characters of degree $\deg\chi$.

Proof. It follows from Nichols-Zoeller Theorem [20]. See also [18, Lemma 2.2.2]. □

Lemma 2.2. *Assume that $\dim H$ is odd and H is of type $(1,n_1;\cdots;d_s,n_s)$ as an algebra. Then d_i is odd and n_i is even for all $2\le i\le s$.*

Proof. It follows from [10, Theorem 5] that d_i is odd.

If there exists $i\in\{2,\cdots,s\}$ such that n_i is odd, then there is at least one irreducible character of degree d_i such that it is self-dual. This contradicts [10, Theorem 4]. □

Lemma 2.3. *Assume that $\dim H$ is odd. If H has a simple module of dimension 3, then 3 divides the order of $G(H^*)$. In particular, 3 divides $\dim H$.*

Proof. Let χ_3 be an irreducible character of degree 3. By Lemma 2.2, H does not have irreducible characters of even degree. Therefore, if $G[\chi_3]$ is trivial then $\chi_3\chi_3^*=\varepsilon+\chi_3'+\chi_5$ for some $\chi_3'\in X_3,\chi_5\in X_5$. Since $\chi_3\chi_3^*$ is self-dual, χ_3' and χ_5 are self-dual. It contradicts the assumption and [10, Theorem 4]. Hence, $G[\chi]$ is not trivial for every $\chi\in X_3$. By Lemma 2.1 (1), the order of $G[\chi]$ is 3 or 9. Thus, 3 divides $|G(H^*)|$ since $G[\chi]$ is a subgroup of $G(H^*)$ for every $\chi\in X_3$.

The second statement can be obtained by the Nichols–Zoeller Theorem. □

Remark 2.4. *The above lemma has appeared in* [2, Corollary] *and* [11, Theorem 4.4]*, respectively. In the first paper, Burciu does not assume that the characteristic of the base field is zero, but adds the assumption that H has no even-dimensional simple modules. Accordingly, his proof is rather different from ours. The author learned the result in the second paper after he finished this paper. Our proof here is slightly different from that in the second paper. So we give the proof for the sake of completeness.*

Corollary 2.5. *Assume that $\dim H$ is odd and H is of type $(1,n;3,m;\cdots)$ as an algebra. If*

(1) H does not have irreducible characters of degree 9, or

(2) there exists a non-trivial subgroup G of $G(H^*)$ such that $G[\chi]=G$ for all $\chi\in X_3$,

then H has a quotient Hopf algebra of dimension $n+9m$.

Proof. Let χ, ψ be irreducible characters of degree 3. By assumption and [4, Lemma 2.5], $\chi\psi$ is not irreducible. If there exists $\chi_5 \in X_5$ such that $m(\chi_5, \chi\psi) > 0$ then $\chi\psi = \chi_5 + \chi_3 + g$ for some $\chi_3 \in X_3$ and $g \in G(H^*)$, by Lemma 2.2. From $m(g, \chi\psi) = m(\chi, g\psi^*) = 1$, we get $\chi = g\psi^*$. Then $\chi\psi = g\psi^*\psi = \chi_5 + \chi_3 + g$ shows that $\psi^*\psi = g^{-1}\chi_5 + g^{-1}\chi_3 + \varepsilon$. This contradicts Lemma 2.3. Similarly, we can show that there does not exist $\chi_7 \in X_7$ such that $m(\chi_7, \chi\psi) > 0$. Therefore, $\chi\psi$ is a sum of irreducible characters of degree 1 or 3. It follows that irreducible characters of degree 1 and 3 span a standard subalgebra of $R(H)$ and H has a quotient Hopf algebra of dimension $n + 9m$. □

Lemma 2.6. *Assume that* $\dim H$ *is odd and H does not have simple modules of dimension* 3 *and* 7*. If H has a simple module of dimension* 5*, then* 5 *divides the order of $G(H^*)$. In particular,* 5 *divides* $\dim H$*.*

Proof. Let χ be an irreducible character of degree 5. By assumption and Lemma 2.2, if $G[\chi]$ is trivial then there are four possible decomposition of $\chi\chi^*$:

$$\chi\chi^* = \varepsilon+\chi_{11}+\chi_{13}; \chi\chi^* = \varepsilon+\chi_9+\chi_{15}; \chi\chi^* = \varepsilon+\chi_5+\chi_{19}; \chi\chi^* = \varepsilon+\chi_5^1+\chi_5^2+\chi_5^3+\chi_9,$$

where χ_i, χ_j^k are irreducible characters of degree i, j. In all cases, there exists at least one irreducible character such that it is self-dual, since $\chi\chi^*$ is self-dual. It contradicts the assumption and [10, Theorem 4]. Therefore, $G[\chi]$ is not trivial for every $\chi \in X_5$. Hence, 5 divides the order of $G(H^*)$ by Lemma 2.1 (1). □

2.2. **Semisolvability.** Let B be a finite-dimensional Hopf algebra over k. A Hopf subalgebra $A \subseteq B$ is called normal if $h_1 A S(h_2) \subseteq A$ and $S(h_1)Ah_2 \subseteq A$, for all $h \in B$. If B does not contain proper normal Hopf subalgebras then it is called simple. The notion of simplicity is self-dual, that is, B is simple if and only if B^* is simple.

The notions of upper and lower semisolvability for finite-dimensional Hopf algebras have been introduced in [15], as generalizations of the notion of solvability for finite groups. By definition, H is called lower semisolvable if there exists a chain of Hopf subalgebras

$$H_{n+1} = k \subseteq H_n \subseteq \cdots \subseteq H_1 = H$$

such that H_{i+1} is a normal Hopf subalgebra of H_i, for all i, and all quotients $H_i/H_iH_{i+1}^+$ are trivial. That is, they are isomorphic to a group algebra or a dual group algebra. Dually, H is called upper semisolvable if there exists a chain of quotient Hopf algebras

$$H_{(0)} = H \xrightarrow{\pi_1} H_{(1)} \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_n} H(n) = k$$

such that $H_{(i-1)}^{co\pi_i} = \{h \in H_{(i-1)} | (id \otimes \pi_i)\Delta(h) = h \otimes 1\}$ is a normal Hopf subalgebra of $H_{(i-1)}$, and all $H_{(i-1)}^{co\pi_i}$ are trivial.

In analogy with the situations for finite groups, it is enough for many applications to know that a Hopf algebra is semisolvable.

By [15, Corollary 3.3], we have that H is upper semisolvable if and only if H^* is lower semisolvable. If this is the case, then H can be obtained from group algebras and their duals by means of (a finite number of) extensions.

2.3. **Radford's biproduct.** Let A be a semisimple Hopf algebra and let ${}^A_A\mathcal{YD}$ denote the braided category of Yetter-Drinfeld modules over A. Let R be a semisimple Yetter-Drinfeld Hopf algebra in ${}^A_A\mathcal{YD}$. Denote by $\rho: R \to A\otimes R$, $\rho(a) = a_{-1}\otimes a_0$, and $\cdot : A\otimes R \to R$, the coaction and action of A on R, respectively. We shall use the notation $\Delta(a) = a^1 \otimes a^2$ and S_R for the comultiplication and the antipode of R, respectively.

Since R is in particular a module algebra over A, we can form the smash product (see [15, Definition 4.1.3]). This is an algebra with underlying vector space $R\otimes A$, multiplication is given by

$$(a\otimes g)(b\otimes h) = a(g_1\cdot b)\otimes g_2h, \text{ for all } g,h\in A, a,b\in R,$$

and unit $1 = 1_R\otimes 1_A$.

Since R is also a comodule coalgebra over A, we can dually form the smash coproduct. This is a coalgebra with underlying vector space $R\otimes A$, comultiplication is given by

$$\Delta(a\otimes g) = a^1\otimes (a^2)_{-1}g_1\otimes (a^2)_0\otimes g_2, \text{ for all } h\in A, a\in R,$$

and counit $\varepsilon_R\otimes\varepsilon_A$.

As observed by D. E. Radford (see [21, Theorem 1]), the Yetter-Drinfeld condition assures that $R\otimes A$ becomes a Hopf algebra with these structures. This Hopf algebra is called the Radford's biproduct of R and A. We denote this Hopf algebra by $R\#A$ and write $a\#g = a\otimes g$ for all $g\in A, a\in R$. Its antipode is given by

$$S(a\#g) = (1\#S(a_{-1}g))(S_R(a_0)\#1), \text{ for all } g\in A, a\in R.$$

A biproduct $R\#A$ as described above is characterized by the following property(see [21, Theorem 3]): suppose that H is a finite-dimensional Hopf algebra endowed with Hopf algebra maps $\iota: A\to H$ and $\pi: H\to A$ such that $\pi\iota: A\to A$ is an isomorphism. Then the subalgebra $R = H^{co\pi}$ has a natural structure of Yetter-Drinfeld Hopf algebra over A such that the multiplication map $R\#A\to H$ induces an isomorphism of Hopf algebras.

The following lemma is a special case of [17, Lemma 4.1.9].

Lemma 2.7. *Let H be a semisimple Hopf algebra of dimension p^2q^2, where p,q are distinct prime numbers. If $gcd(|G(H)|,|G(H^*)|) = p^2$, then $H\cong R\#kG$ is a biproduct, where kG is the group algebra of group G of order p^2, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

3. Semisimple Hopf algebras of dimension p^2q^2

Let p,q be distinct prime numbers with $p<q$. Throughout this section, H will be a semisimple Hopf algebra of dimension p^2q^2, unless otherwise stated. By Nichols-Zoeller Theorem [20], the order of $G(H^*)$ divides $\dim H$. Moreover, $|G(H^*)|\neq 1$ by [6, Proposition 9.9]. By [4, Lemma 1], H is of Frobenius type. Therefore, the dimension of a simple H-module can only be $1,p,p^2$ or q. Let a,b,c be the number of non-isomorphic simple H-modules of dimension p,p^2 and q, respectively. It follows that we have an equation $p^2q^2 = |G(H^*)| + ap^2 + bp^4 + cq^2$. In particular, if $|G(H^*)| = p^2q^2$ then H is a dual group algebra; if $|G(H^*)| = pq^2$ then H is upper semisolvable by the following lemma, which is due to [4, Lemma 2.3].

Lemma 3.1. *If H has a Hopf subalgebra K of dimension pq^2 then H is lower semisolvable.*

The following lemma is a refinement of [4, Lemma 3.4].

Lemma 3.2. *If the order of $G(H^*)$ is q^2 then H is upper semisolvable.*

Proof. If $p = 2$ and $q = 3$ then it is the case discussed in [17, Chapter 8]. Hence, H is upper semisolvable. Throughout the remainder of the proof, we assume that $p \geq 3$.

By Lemma 2.1 (2), if $a \neq 0$ then $ap^2 \geq p^2q^2$, a contradiction. Hence, $a = 0$. Similarly, $b = 0$. If follows that H is of type $(1, q^2; q, p^2 - 1)$ as an algebra.

The group $G(H^*)$ acts by left multiplication on the set X_q. The set X_q is a union of orbits which have length $1, q$ or q^2. Since $q > p \geq 3$, q does not divides $p^2 - 1$. Therefore, there exists one orbit with length 1. That is, there exists an irreducible character $\chi_q \in X_q$ such that $G[\chi_q] = G(H^*)$. This means that $g\chi_q = \chi_q = \chi_q g$ for all $g \in G(H^*)$.

Let C be a q^2-dimensional simple subcoalgebra of H^*, corresponding to χ_q. Then $gC = C = Cg$ for all $g \in G(H^*)$. By [17, Proposition 3.2.6], $G(H^*)$ is normal in $k[C]$, where $k[C]$ denotes the subalgebra generated by C. It is a Hopf subalgebra of H^* containing $G(H^*)$. Counting dimension, we know $\dim k[C] \geq 2q^2$. Since $\dim k[C]$ divides $\dim H$, we know $\dim k[C] = pq^2$ or p^2q^2. If $\dim k[C] = pq^2$ then Lemma 3.1 shows that H^* is lower semisolvable. If $\dim k[C] = p^2q^2$ then $k[C] = H^*$. Since $kG(H^*)$ is a group algebra and the quotient $H^*/H^*(kG(H^*))^+$ is trivial (see [13]), H^* is lower semisolvable. Hence, H is upper semisolvable. This completes the proof. □

Theorem 3.3. *If $q > p^2$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order p^2, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

Proof. By [1, Proposition 1.1], H has a quotient Hopf algebra $\overline{H}$ of dimension $|G(H^*)| + ap^2 + bp^4$. In particular, $|G(H^*)|$ divides $\dim\overline{H}$ and $|G(H^*)| + ap^2 + bp^4$ divides $\dim H$.

We first prove that the order of $G(H^*)$ can not be q. Suppose on the contrary that $|G(H^*)| = q$. We first note that $c \neq 0$, since otherwise we get the contradiction $p^2 \mid q$. Since q divides $\dim\overline{H}$ and $c \neq 0$, we have that $\dim\overline{H} < p^2q^2$. Therefore, $\dim\overline{H} = q, pq, p^2q, pq^2$ or q^2. If $\dim\overline{H} = q^2$ then $(\overline{H})^* \subseteq kG(H^*)$ by [13]. It is impossible since $q^2 = \dim\overline{H}$ does not divide $|G(H^*)| = q$. If $\dim\overline{H} = q, pq$ or p^2q then we have $p^2q^2 = q + cq^2$, $p^2q^2 = pq + cq^2$ or $p^2q^2 = p^2q + cq^2$. They all impossible. Hence, $\dim\overline{H} = p^2q$. That is $q + ap^2 + bp^4 = pq^2$. It is impossible, too.

We then prove that if $|G(H^*)| = p$ or pq then H is upper semisolvable. We first note that $c \neq 0$, since otherwise we get the contradiction $p^2 \mid p$. Then $p \mid \dim\overline{H}$ and $\dim\overline{H} < p^2q^2$. Therefore $\dim\overline{H} = p, pq, p^2q, pq^2$ or p^2. Moreover, $\dim\overline{H} \neq p^2$, since otherwise $(\overline{H})^* \subseteq kG(H^*)$ by [13], but $p^2 = \dim\overline{H}$ does not divide $|G(H^*)| = p$ or pq. The possibilities $\dim\overline{H} = p, pq$ or p^2q lead, respectively to the contradictions $p^2q^2 = p + cq^2$, $p^2q^2 = pq + cq^2$ and $p^2q^2 = p^2q + cq^2$. Hence these are also discarded, and therefore $\dim\overline{H} = pq^2$. This implies that H is upper semisolvable, by Lemma 3.1.

Finally, the theorem follows from Lemma 2.7, 3.1 and 3.2. □

As an immediate consequence of Theorem 3.3, we have the following corollary.

Corollary 3.4. *If $p^2 < q$ and H is simple as a Hopf algebra then H is isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order p^2, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

In fact, examples of nontrivial semisimple Hopf algebras of dimension p^2q^2 which are Radford's biproducts in such a way, and are simple as Hopf algebras do exists. A construction of such examples as twisting deformations of certain groups appears in [7, Remark 4.6].

4. Applications

4.1. **Semisimple Hopf algebras of dimension $9q^2$.** In this subsection, we shall prove the following theorem.

Theorem 4.1. *If H is a semisimple Hopf algebra of dimension $9q^2$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

By Theorem 3.3, it suffices to consider the case $q = 5$ and 7.

Lemma 4.2. *If $q = 5$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension 25.*

Proof. By Lemma 2.1, 2.2 and 2.3, if $\dim H = 3^2 \times 5^2$ then H is of one of the following types as an algebra:

$$(1, 25; 5, 8), (1, 75; 5, 6), (1, 3; 3, 8; 5, 6), (1, 9; 3, 6; 9, 2), (1, 9; 3, 24), (1, 45; 3, 20).$$

If H is of type $(1, 25; 5, 8)$ as an algebra then Lemma 3.2 shows that H is upper semisolvable. If H is of type $(1, 75; 5, 6)$ as an algebra then Lemma 3.1 shows that H is upper semisolvable. If H is of type $(1, 3; 3, 8; 5, 6)$ as an algebra then Corollary 2.5 shows that H has a quotient Hopf algebra of dimension 75. Hence, Lemma 3.1 shows that H is upper semisolvable. The lemma then follows from Lemma 2.7. $\square$

Remark 4.3. *The computation in the proof of Lemma 4.2 is partly handled by a computer. For example, it is easy to write a computer program by which one finds out all non-negative integers n_1, n_2, n_3, n_4 such that $225 = n_1 + 9n_2 + 81n_3 + 25n_4$, and then one can eliminate those which can not be algebra types of H by using Lemma 2.1, 2.2 and 2.3. The computations in the followings are handled similarly.*

Lemma 4.4. *If $q = 7$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension 49.*

Proof. By Lemma 2.1, 2.2 and 2.3, if $\dim H = 3^2 \times 7^2$ then H is of one of the following types as an algebra:

$$(1, 3; 3, 14; 5, 6; 9, 2), (1, 3; 3, 32; 5, 6), (1, 3; 3, 16; 7, 6), (1, 21; 3, 14; 7, 6),$$

$$(1, 49; 7, 8), (1, 147; 7, 6), (1, 9; 3, 12; 9, 4), (1, 9; 3, 30; 9, 2), (1, 9; 3, 48), (1, 63; 3, 42).$$

Corollary 2.5 shows that H can not be of type $(1, 3; 3, 14; 5, 6; 9, 2), (1, 3; 3, 32; 5, 6)$ as an algebra, since it contradicts Nichols-Zoeller Theorem. The lemma then follows from a similar argument as in Lemma 4.2. $\square$

Corollary 4.5. *If H is a semisimple Hopf algebra of dimension $9q^2$ and is simple as a Hopf algebra then H is isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 9, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

4.2. **Semisimple Hopf algebras of dimension $25q^2$.** In this subsection, we shall prove the following theorem.

Theorem 4.6. *If H is a semisimple Hopf algebra of dimension $25q^2$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 25, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

By Theorem 3.3, it suffices to consider the case $7 \leq q \leq 23$.

Lemma 4.7. *If $q = 7$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 25, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension 49.*

Proof. By Lemma 2.1, 2.2 and 2.6, if $\dim H = 5^2 \times 7^2$ then H is of one of the following types as an algebra:

$$(1,35;5,28;7,10),(1,49;7,24),(1,245;7,20),(1,175;5,42),(1,25;5,48).$$

We shall prove that H can not be of type $(1,35;5,28;7,10)$ as an algebra. The lemma then will follow from Lemma 2.7, 3.1 and 3.2.

Suppose on the contrary that H is of type $(1,35;5,28;7,10)$ as an algebra. The group $G(H^*)$ acts by left multiplication on the set X_5. The set X_5 is a union of orbits which have length 1, 5 or 7. By Lemma 2.1 (1), $G[\chi]$ is a proper subgroup of $G(H^*)$ for every $\chi \in X_5$. Hence, there does not exist orbits with length 1. Accordingly, every orbit has length 7 and the order of $G[\chi]$ is 5 for every $\chi \in X_5$. In particular, the decomposition of $\chi\chi^*$ does not contain irreducible characters of degree 7.

Let χ', χ be distinct irreducible characters of degree 5. Suppose that there exists $\chi_7 \in X_7$ such that $m(\chi_7, \chi'\chi^*) > 0$. Then there must exist $\varepsilon \neq g \in G(H^*)$ such that $m(g, \chi'\chi^*) = 1$. From this observation, we know $\chi' = g\chi$ and $\chi'\chi^* = g\chi\chi^*$. Since $\chi\chi^*$ does not contain irreducible characters of degree 7, $\chi'\chi^*$ does not contain such characters, too. This contradicts the assumption. Therefore, $\chi'\chi^*$ is a sum of irreducible characters of degree 1 or 5. It follows that $G(H^*) \cup X_5$ spans a standard subalgebra of $R(H)$, and H has a quotient Hopf algebra of dimension 735. This contradicts the Nichols-Zoeller Theorem [20]. □

Lemma 4.8. *Let H be a semisimple Hopf algebra of dimension $25q^2$, where $q = 11, 17, 19$. If $|G(H^*)| = 5$ or $5q$ then H has a quotient Hopf algebra of dimension $|G(H^*)| + 25a$, where a is the cardinal number of X_5.*

Proof. In fact, it can be checked directly that $G[\chi] = 5$ for every $\chi \in X_5$. Then the lemma follows from a similar argument as in the proof of Lemma 4.7. □

Lemma 4.9. *If $q = 11$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 25, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension 121.*

Proof. By Lemma 2.1, 2.2 and 2.6, if $\dim H = 5^2 \times 11^2$ then H is of one of the following types as an algebra:

$$(1,5;5,24;11,20), (1,55;5,22;11,20), (1,121;11,24),$$

$$(1,605;11,20), (1,275;5,110), (1,25;5,20;25,4), (1,25;5,70;25,2), (1,25;5,120).$$

By Lemma 4.8, if H is of type $(1,5;5,24;11,20)$ or $(1,55;5,22;11,20)$ as an algebra then H has a quotient Hopf algebra of dimension 605. Then H is upper semisolvable by Lemma 3.1. The lemma then follows from Lemma 2.7, 3.1 and 3.2. □

Lemma 4.10. *If $q = 13$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order* 25*, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension* 169*.*

Proof. By Lemma 2.1, 2.2 and 2.6, if $\dim H = 5^2 \times 13^2$ then H is of one of the following types as an algebra:

$$(1,169;13,24), (1,845;13,20), (1,325;5,156),$$

$$(1,25;5,18;25,6), (1,25;5,168), (1,25;5,68;25,4), (1,25;5,118;25,2).$$

The lemma then follows directly from Lemma 2.7, 3.1 and 3.2. □

Lemma 4.11. *If $q = 17$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order* 25*, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension* 289*.*

Proof. By Lemma 2.1, 2.2 and 2.6, if $\dim H = 5^2 \times 17^2$ then H is of one of the following types as an algebra:

$$(1,85;5,170;17,10),$$

$$(1,289;17,24), (1,1445;17,20), (1,425;5,272), (1,25;5,288), (1,25;5,238;25,2)$$

$$(1,25;5,38;25,10), (1,25;5,88;25,8), (1,25;5,138;25,6), (1,25;5,188;25,4).$$

By Lemma 4.8, if H is of type $(1,85;5,170;17,10)$ as an algebra then H has a quotient Hopf algebra of dimension 4335. This contradicts the Nichols-Zoeller Theorem [20]. The lemma then follows from Lemma 2.7, 3.1 and 3.2. □

Lemma 4.12. *If $q = 19$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order* 25*, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension* 361*.*

Proof. By Lemma 2.1, 2.2 and 2.6, if $\dim H = 5^2 \times 19^2$ then H is of one of the following types as an algebra:

$$(1,5;5,22;19,20;25,2), (1,5;5,72;19,20),$$

$$(1,361;19,24), (1,1805;19,20), (1,475;5,342),$$

$$(1,25;5,10;25,14), (1,25;5,60;25,12), (1,25;5,110;25,10), (1,25;5,360),$$

$$(1,25;5,160;25,8), (1,25;5,210;25,6), (1,25;5,260;25,4), (1,25;5,310;25,2).$$

By Lemma 4.8, if H is of type $(1,5;5,22;19,20;25,2)$ as an algebra then H has a quotient Hopf algebra of dimension 555. This contradicts the Nichols-Zoeller Theorem [20]. Again by Lemma 4.8, if H is of type $(1,5;5,72;19,20)$ as an algebra then H has a quotient Hopf algebra of dimension 1805. Then H is upper semisolvable by Lemma 3.1. The lemma then follows from Lemma 2.7, 3.1 and 3.2. □

Lemma 4.13. *If $q = 23$ then H is either semisolvable or isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 25, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension 529.*

Proof. By Lemma 2.1, 2.2 and 2.6, if $\dim H = 5^2 \times 23^2$ then H is of one of the following types as an algebra:

$$(1, 529; 23, 24), (1, 2645; 23, 20), (1, 575; 5, 506),$$

$$(1, 25; 5, 28; 25, 20), (1, 25; 5, 528), (1, 25; 5, 478; 25, 2)$$

$$(1, 25; 5, 278; 25, 10), (1, 25; 5, 328; 25, 8), (1, 25; 5, 378; 25, 6), (1, 25; 5, 428; 25, 4),$$

$$(1, 25; 5, 78; 25, 18), (1, 25; 5, 128; 25, 16), (1, 25; 5, 178; 25, 14), (1, 25; 5, 228; 25, 12).$$

The lemma then follows directly from Lemma 2.7, 3.1 and 3.2. □

Corollary 4.14. *If H is a semisimple Hopf algebra of dimension $25q^2$ and is simple as a Hopf algebra then H is isomorphic to a Radford's biproduct $R\#kG$, where kG is the group algebra of group G of order 25, R is a semisimple Yetter-Drinfeld Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$ of dimension q^2.*

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